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Semigroup of Nonexpansive Mappings on a Hilbert Space

ANTHONY TO-MING LAU*

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada

Submitted by K. Fan

1. INTRODUCTION

Let S be a *semitopological semigroup*, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from S to S are continuous. Let H be a Hilbert space and $\mathscr{S} = \{T_s; s \in S\}$ be a continuous representation of S as nonexpansive mappings on a closed convex subset C of H into C, i.e., $T_{ab}(x) = T_a T_b(x)$, $a, b \in S$, $x \in C$, and the mapping $(s, x) \to T_s(x)$ from $S \times C$ into C is continuous when $S \times C$ has the product topology. Let $F(\mathscr{S})$ denote the set $\{x \in C; T_s(x) = x$ for all $s \in S\}$ of common fixed points of $\mathscr{S}$ in C. Then, as is well known, $F(\mathscr{S})$ (possibly empty) is a closed convex subset of C (see [4]).

In Section 2 of this paper, we shall be concerned with the following problem: Let $x \in C$. Does $\{T_s(x); s \in S\}$ converge weakly to a common fixed point z of $\mathscr{S}$ in C in the sense that for each $y \in H$ and each $\varepsilon > 0$, there exists $a \in S$ such that $|\langle T_{sa}(x) - z, y \rangle| < \varepsilon$ for all $s \in S$? When T is a nonexpansive mapping of C into C and $\mathscr{S} = \{T^n; n = 1, 2, 3,...\}$, this problem is equivalent to that of the weak convergence of the sequence $\{T^n(x); n = 1, 2,...\}$ to a fixed point of T considered by Z. Opial in [12] and more recently by A. Pazy in [13].

A semitopological semigroup S is *right reversible* if any two closed left ideals of S has nonvoid intersection. In this case, $(S, \preccurlyeq)$ is a directed system when the binary relation "$\preccurlyeq$" on S is defined by $a \preccurlyeq b$ if and only if $\{a\} \cup \overline{Sa} \supseteq \{b\} \cup \overline{Sb}$, $a, b \in S$. Also, weak convergence of $\{T_s(x); s \in S\}$ as defined earlier is equivalent to weak convergence of the net $\{T_s(x); s \in S\}$. Examples of right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right

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amenable as discrete semigroups (see [5, p. 335]). Left reversibility of S is defined similarly.

We prove (Theorem 2.3), among other things, that if S is right reversible, $F(\mathscr{S})$ is nonempty and the net $\|T_{gs}(x) - T_s(x)\| \to 0$ for each fixed g in a generating set of S, then the net $\{T_s(x); s \in S\}$ converges weakly to an element in $F(\mathscr{S})$. We also prove that (Proposition 2.4) this weak limit of $\{T_s(x); s \in S\}$ agrees with the norm limit of $\{P(T_s(x)); s \in S\}$ (which always exist) when P is the metric projection on $F(\mathscr{S})$.

Our results are well known when $\mathscr{S} = \{T^n; n \in N\}$ and T is a nonexpansive mapping from C into C (see Belluce and Kirk [1], Opial [12], and Pazy [13]). Simple example (say, rotation of the plane about the origin) shows that if T is a nonexpansive mapping from C into C and $x \in C$, the sequence $\{T^n(x); n = 1,...,\}$ need not converge weakly.

Recently, Lim [8] proved that if K is a weak*-closed convex nonempty subset of l_1 and S is left reversible, then for any continuous representation $\mathscr{S} = \{T_s; s \in S\}$ of S as nonexpansive mappings from K into K such that $\{T_{as}(x); s \in \mathscr{S}\}$ is bounded for some $x \in K$ and $a \in S$, then $\mathscr{S}$ has a common fixed point in K. In Section 3, we shall prove some variants of Lim's result for closed convex subset C of a Hilbert space H. We prove (Theorem 3.3) that if the space of almost periodic function on S has a left invariant mean and if there exist $x \in C$ with relatively compact orbit, then C contains a common fixed point for $\mathscr{S}$. We also prove that (Theorem 3.5) the same conclusion also holds if the space RUC(S) of right uniformly continuous functions on S has a left invariant mean and there exist $x \in C$ with bounded orbit.

Note that if C is bounded and S is left reversible, it follows from a result of Holmes and Lau [5, Theorem 1] and Belluce and Kirk [2, Theorem 4.1] (see also Lim [7, Theorem 3]) that any continuous representation of S as nonexpansive mappings from C into C has a common fixed point in C.

2. Weak Convergence of $\{T_s(x); s \in S\}$

Throughout the paper, unless otherwise specified, S denotes a semitopological semigroup and $\mathscr{S} = \{T_a; a \in S\}$ a continuous representation of S as nonexpansive mappings from a nonempty closed convex subset C of a Hilbert space H into C. If S is right reversible and S is directed as in Section 1, then for each $x \in C$, let $\omega(x)$ denote the set of all weak limit points of subnets of the net $\{T_a(x); a \in S\}$.

We will need in our proofs the following modification of Opial's condition (with the same proof) for bounded nets in a Hilbert space [12, Lemma 1]:

LEMMA 2.1. *Let H be a Hilbert space and let $\{x_\alpha\}$ be a bounded net in H converging weakly to x_0. Then for any $x \in H$, $x \neq x_0$,*

$$\liminf \|x_\alpha - x\| > \liminf \|x_\alpha - x_0\|.$$

Here $\lim_\alpha \inf\{s_\alpha\}$ *of a bounded net of real number* $\{s_\alpha\}$ *is the limit of the increasing net* $\{t_\beta\}$, *where* $t_\beta = \inf\{s_\alpha; \alpha \geqslant \beta\}$.

The following lemma is proved by Pazy [13, Theorem 3] for the semigroup $\mathscr{S} = \{T^n; n = 1, 2, \ldots\}$. However, his proof, dependent heavily on properties of the metric projection of H onto $F(\mathscr{S})$, is completely different from ours (see remark after Theorem 2.3).

LEMMA 2.2. *Assume that S is right reversible and $x \in C$.*

(a) *If $F(\mathscr{S})$ is nonempty and $\omega(x) \subseteq F(\mathscr{S})$, then the net $\{T_a(x); a \in S\}$ converges weakly to some $y \in F(\mathscr{S})$.*

(b) *Suppose the net $\{T_a(x); a \in S\}$ converges weakly to some $y \in C$. Then $y \in F(\mathscr{S})$ if and only if $\{T_a(x); a \in S\}$ is bounded.*

Proof. (a) If $y \in F(\mathscr{S})$, then the net $\{\|T_a x - y\|; a \in S\}$ is bounded. Also if $b \geqslant a$, $b \in \overline{Sa}$ (say), let $\{s_\alpha\}$ be a net in S such that $s_\alpha a \to b$. Then, for each α,

$$\|T_{s_\alpha a} x - y\| = \|T_{s_\alpha}(T_a x) - T_{s_\alpha}(y)\| \leqslant \|T_a(x) - y\|.$$

Hence $\|T_b x - y\| \leqslant \|T_a x - y\|$. Consequently, the limit $\lim_a \|T_a x - y\|$ exists and is finite for each $y \in F(\mathscr{S})$. Since the net $\{T_a(x); a \in S\}$ is bounded, it must contain a subnet $\{T_{a_\gamma}(x)\}$ which converges weakly to some $z \in C$. By assumption, $z \in F(\mathscr{S})$. Suppose $\{T_a(x); a \in S\}$ does not converge weakly to z, then there exists another subnet $\{T_{a_\beta}(x)\}$ which converges weakly to some $u \in F(\mathscr{S})$, $z \neq u$. Now by Lemma 2.1,

$$\lim_\alpha \|T_{a_\gamma}(x) - z\| < \lim_\alpha \|T_{a_\gamma}(x) - u\| = \lim_a \|T_a(x) - u\|$$

and

$$\lim_a \|T_a(x) - u\| = \lim_\beta \|T_{a_\beta}(x) - u\| < \lim_\beta \|T_{a_\beta}(x) - z\|$$

which is impossible since

$$\lim_\gamma \|T_{a_\gamma}(x) - z\| = \lim_\beta \|T_{a_\beta}(x) - z\|$$

by convergence of the net $\{\|T_a(x) - z\|; a \in S\}$.

(b) If $F(\mathscr{S})$ is nonempty and $z \in F(\mathscr{S})$, then $\|T_s x - z\| \leqslant \|x - z\|$ for each $s \in S$. Hence $\{T_s(x); s \in S\}$ is bounded.

Conversely if $\{T_s(x); s \in S\}$ is bounded, let $a \in S$. If $T_a(y) \neq y$, then by Lemma 2.1

$$\rho = \liminf_s \{\|T_s(x) - T_a(y)\|; s \in S\} > \liminf_s \{\|T_s(x) - y\|; s \in S\}.$$

Given $\varepsilon > 0$, choose $b \in S$ such that

$$\inf\{\|T_s(x) - T_a(y)\|; s \geqslant b\} > \rho - \varepsilon.$$

In particular

$$\|T_{as}(x) - T_a(y)\| > \rho - \varepsilon$$

for all $s \geqslant b$. Since $\varepsilon > 0$ is arbitrary, we have $\lim_s \inf\{\|T_{as}(x) - T_a(y)\|\} \geqslant \rho$. On the other hand, $\|T_{as}(x) - T_a(y)\| \leqslant \|T_s(x) - y\|$ for all $s \in S$. Hence

$$\liminf_s \{\|T_{as}(x) - T_a(y)\|\} \leqslant \liminf_s \{\|T_s(x) - y\|; s \in S\} < \rho$$

which is impossible. Hence $T_a(y) = y$.

A subset G of S is called a *generating set* if elements of the form $g_1 g_2 \cdots g_n$, $g_1, g_2, ..., g_n \in G$, is dense in S.

THEOREM 2.3. *Assume that S is right reversible and $x \in C$. If $F(\mathscr{S})$ is nonempty and $\|T_{ga}(x) - T_a(x)\| \to 0$ for all g in a generating set G of S, then the net $\{T_a(x); a \in S\}$ converges weakly to an element of $F(\mathscr{S})$.*

Proof. By Lemma 2.2, it suffices to show that $\omega(x) \subseteq F(\mathscr{S})$. Let $\{T_{a_\alpha}(x)\}$ be a subnet of $\{T_a(x); a \in S\}$ converging weakly to some $y \in C$. Let $g \in G$. If $T_g(y) \neq y$, then by Lemma 2.1

$$\liminf_\alpha \|T_{a_\alpha}(x) - y\| < \liminf_\alpha \|T_{a_\alpha}(x) - T_g(y)\|. \tag{1}$$

On the other hand

$$\begin{aligned}\|T_{a_\alpha}(x) - T_g(y)\| &\leqslant \|T_{a_\alpha}(x) - T_{ga_\alpha}(x)\| + \|T_{ga_\alpha}(x) - T_g(y)\| \\ &\leqslant \|T_{a_\alpha}(x) - T_{ga_\alpha}(x)\| + \|T_{a_\alpha}(x) - y\|.\end{aligned}$$

In particular

$$\liminf_\alpha \|T_{a_\alpha}(x) - T_g(y)\| \leqslant \liminf_\alpha \|T_{a_\alpha}(x) - y\|$$

since $\lim_\alpha \|T_{a_\gamma}(x) - T_{ga_\gamma}(x)\| = 0$. This contradicts (1). Hence $T_g(y) = y$. Since $g \in G$ is arbitrary, it follows that $y \in F(\mathcal{S})$.

Remark. Theorem 2.3 is due to Opial [12, Theorem 1] when $\mathcal{S} = \{T^n; n \in N\}$ and $G = \{1\}$ and T is a nonexpansive mapping from C into C. Since Lemma 2.1 is valid for any uniformly convex Banach spacing having a weakly continuous duality map (see [12, Lemma 3]) so are Lemma 2.2 and Theorem 2.3. If $1 < p < \infty$, then the Banach spaces l_p are uniformly convex with a weakly continuous duality map (see [3]).

The next result is due to Pazy for $\mathcal{S} = \{T^n; n \in N\}$ [13, Lemma 2]. The proof follows an idea of Pazy there.

PROPOSITION 2.4. *Assume that S is right reversible and $F(\mathcal{S})$ is non-empty. Let P be the metric projection of H onto $F(\mathcal{S})$. Then for each $x \in C$, the net $\{PT_a(x); a \in S\}$ converges in norm to some $z \in F(\mathcal{S})$. Furthermore if the net $\{PT_a(x); a \in S\}$ converges weakly to some $y \in F(\mathcal{S})$, then $y = z$.*

Proof. Observe that

$$\|P(T_a x) - T_a x\| \leqslant \|P(T_b x) - T_a x\| \tag{2}$$

for any $a, b \in S$. If $a \geqslant b$ and $a \neq b$, let $s_\alpha b$ be a net converging to a. Then for each α,

$$\|P(T_b x) - T_{s_\alpha b} x\| = \|T_{s_\alpha} P(T_b x) - T_{s_\alpha}(T_b x)\| \leqslant \|P(T_b x) - T_b x\| \tag{3}$$

i.e., $\|P(T_b x) - T_a x\| \leqslant \|P(T_b x) - T_b x\|$ for $a \geqslant b$. Hence if $a \geqslant b$, then

$$\|P(T_a x) - T_a x\| \leqslant \|P(T_b x) - T_b x\| \tag{4}$$

by (2). Let $u \in F(\mathcal{S})$ and $v \in H$. We have

$$\|Pv - u\|^2 \leqslant \|v - u\|^2 - \|Pv - v\|^2$$

by property of P. Put $v = T_a x$ and $u = PT_b x$, $a \geqslant b$, we obtain

$$\begin{aligned} \|P(T_a x) - P(T_b x)\|^2 &\leqslant \|T_a x - P(T_b x)\|^2 - \|P(T_a x) - T_a x\|^2 \\ &\leqslant \|T_b x - P(T_b x)\|^2 - \|P(T_a x) - T_a x\|^2 \end{aligned}$$

by (3). It follows from (4) that $\{P(T_a x); x \in S\}$ is a norm Cauchy net in H. Hence it must converge to some $z \in F(\mathcal{S})$.

If $\{T_a x; a \in S\}$ converges weakly to some $y \in F(\mathcal{S})$, then, by property of P, we have

$$\operatorname{Re}\langle T_a x - P(T_a x), u - PT_a x\rangle \leqslant 0$$

for all $u \in F(\mathscr{S})$, $a \in S$. Hence

$$\operatorname{Re}\langle y - z, u - z\rangle \leqslant 0$$

for all $u \in F(\mathscr{S})$. In particular $y = z$.

Remark. If S is left reversible and C is bounded, then the assumption that $F(\mathscr{S})$ be nonempty can be omitted in Lemma 2.2, Theorem 2.3 and Proposition 2.4 since in this case, $F(\mathscr{S})$ is always nonempty by [5, Theorem 1] and [2, Theorem 4.1].

3. Semigroup of Nonexpansive Mappings with Bounded Orbit

If $f \in C(S)$, $a \in S$, define $(l_a f)() = f(as)$ and $(r_a f)(s) = f(sa)$ for all $s \in S$. If $X \subseteq C(S)$ is a closed subspace of $C(S)$ containing constants and $l_a(X) \subseteq X$ for all $a \in S$, then $m \in X^*$ is called a *left invariant mean* if $\|m\| = m(1) = 1$ and $m(l_a f) = m(f)$ for all $a \in S$, $f \in X$.

Let $\mathrm{AP}(S)$ denote all *continuous almost periodic functions* on S, i.e., all $f \in C(S)$ such that $\{r_a f; a \in S\}$ is relatively compact in the norm topology of $C(S)$. Then, as is well known, $\mathrm{AP}(S)$ is a closed subalgebra of $C(S)$ invariant under translations (left and right).

Lemma 3.1. *If $x \in C$ with relatively compact orbit and $y \in H$, then the following functions are in* $\mathrm{AP}(S)$:

(i) $h(s) = \langle y, T_s x\rangle$,

(ii) $k(s) = \|y - T_s x\|$.

Proof. (i) follows from Lemma 3.1 in [6].

(ii) It is clear that $k \in C(S)$. To see that k is almost periodic, for each $z \in C$, let

$$k_z(s) = \|y - T_s z\|.$$

Then $r_a k_x(s) = k_w(s)$ where $w = T_a x$. Let $\tau\colon a \to k_z$. If we can show that τ is continuous when $C(S)$ has the sup norm topology, then $\tau(\overline{0(x)})$ is a compact subset of $C(S)$ containing $\{r_a k; a \in S\}$. In particular, $k \in \mathrm{AP}(S)$.

To see that τ is continuous, let $\{z_n\}$ be a sequence in C, $z_n \to z$, then

$$\begin{aligned}|\tau(z_n)(s) - \tau(z)(s)| &= |\,\|y - T_s z_n\| - \|y - T_s z\|\,|\\ &\leqslant |\,\|T_s z_n - T_s z\|\,| \leqslant \|z_n - z\|\end{aligned}$$

by nonexpansiveness of T_s, $s \in S$. Hence $\|\tau(z_n) - \tau(z)\| \to 0$ also.

Let $x \in C$ with relatively compact orbit, and $m \in \mathrm{AP}(S)^*$ be a mean, i.e., $\|m\| = m(1) = 1$. Then $\phi(y) = m(h_y)$, where $h_y(s) = \langle y, T_s x \rangle$ defines a bounded linear functional on H (by Lemma 3.1). Hence, by the Riesz representation theorem, there exists $z \in H$ such that $\phi(y) = \langle y, z \rangle$ for all $y \in H$. Since m is the weak* limit of finite means of the form $\sum_{i=1}^n \lambda_i \delta_{a_i}$, $\lambda_i \geqslant 0$, $\sum_{i=1}^n \lambda_i = 1$, $a_i \in S$, where $\delta_a(f) = f(a)$, $f \in \mathrm{AP}(S)$, and C is weakly closed, it follows that $z \in C$. Write $T_m(x) = z$.

LEMMA 3.2. *If m is a left invariant mean on* AP(S), *then $T_m(x)$ is a common fixed point for $\{T_a; a \in S\}$.*

Proof. Let $a \in S$ and $z = T_m(x)$. For each $s \in S$,

$$\|T_a z - T_{as} x\| = \|T_a z - T_a(T_s(x))\| \leqslant \|z - T_s(x)\|.$$

Hence

$$\begin{aligned} \|T_a z - T_{as} x\|^2 &\leqslant \|z - T_s x\|^2 \\ &= \|(z - T_a z) + (T_a z - T_s x)\|^2 \qquad (5) \\ &= \|(z - T_a z)\|^2 + \|T_a z - T_s x\|^2 + 2\,\mathrm{Re}\langle z - T_a z, T_a z - T_s x\rangle. \end{aligned}$$

Now for $w \in H$, the function $k(s) = \|w - T_s x\|^2$ is in AP(S) by Lemma 3.1. Hence if $w = T_a z$, then $l_a k(s) = k(as) = \|T_a z - T_{sa} x\|^2$ is in AP(S). As in Rodé [14, Lemma 1], we apply the left invariant mean "m" to the inequality (5). Then $0 \leqslant \|z - T_a z\|^2 + 2\,\mathrm{Re}\langle z - T_a z, T_a z - z\rangle$ or $0 \leqslant -\|z - T_a z\|^2$. Hence $z = T_a z$. Since $a \in S$ is arbitrary, $z \in F(\mathscr{S})$.

THEOREM 3.3. *If $AP(S)$ has a left invariant mean, $x \in C$ such that $\{T_s(x); s \in S\}$ is relatively compact, then C contains a common fixed point for $\mathscr{S}$.*

This follows from Lemma 3.2.

Remark. If S is a left reversible semitopological semigroup, then $AP(S)$ has a left invariant mean. However there exists semitopological semigroup S which is not left reversible but $C(S)$ has a left invariant mean (for details see [5] and [6]).

Let RUC(S) denote the space of bounded right uniformly continuous functions on S, i.e., all $f \in C(S)$ such that the function $a \to r_a f$ is continuous when $C(S)$ has the norm topology. Then RUC(S) is a closed translation invariant subalgebra of $C(S)$ containing constants. (See [11] for more details.) Proof of the following lemma is routine. We omit the details.

LEMMA 3.4. *If $x \in C$ with bounded orbit and $y \in H$, then the following functions are in* RUC(S):

(i) $h(s) = \langle y, T_s x \rangle$,

(ii) $k(s) = \| y - T_s x \|$.

THEOREM 3.5. *If* RUC(S) *has a left invariant mean and there exist* $x \in C$ *such that* $\{T_s(x); s \in S\}$ *is bounded, then* $\mathscr{S}$ *has a common fixed point in* C.

Proof. This follows from Lemma 3.4 and an argument similar to that of Lemma 3.2.

Remark. (a) A function $f \in C(S)$ is weakly almost periodic if $\{r_a f; a \in S\}$ is relatively compact in the weak topology of $C(S)$. Let WAP(S) denote the space of weakly almost periodic functions on S. Then, in general, $\mathrm{AP}(S) \subseteq \mathrm{WAP}(S) \cap \mathrm{RUC}(S)$ but $\mathrm{WAP}(S) \not\subset \mathrm{RUC}(S)$ and $\mathrm{RUC}(S) \not\subset \mathrm{WAP}(S)$ unless S is algebraically a group; in this case $\mathrm{WAP}(S) \subseteq \mathrm{RUC}(S)$ (see [9, Lemma 4.1] and [10]).

PROBLEM 1. Are the functions h and k defined in Lemma 3.4 in WAP(S)? Is Theorem 3.5 true with RUC(S) replaced by WAP(S)?

(b) Lemma 3.2 implies that if S is left reversible (or more generally if AP(S) has a left invariant mean), and $\mathscr{S} = \{T_s; s \in S\}$ is a continuous representation of S as nonexpansive mapping from a closed convex subset C of a Hilbert space into C such that $T_s(A) \subseteq A$ for some nonempty compact subset A of C, then there exists z in the closed convex hull of A such that $T_s z = z$ for all $s \in S$.

PROBLEM 2. Is this true for arbitrary Banach spaces? (Compare with [6, Theorem 4.1].)

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